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A criterion for ample vector bundles over a curve in positive characteristic

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Abstract

Let X be a smooth projective curve defined over an algebraically closed field of positive characteristic. We give a necessary and sufficient condition for a vector bundle over X to be ample. This generalizes a criterion given by Lange in [Math. Ann. 238 (1978) 193–202] for a rank two vector bundle over X to be ample.

MSC: 14H60; 14F05

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1. Introduction

Let X be a connected smooth projective curve defined over an algebraically closed field k of characteristic zero. In [3, p. 83, Corollary 7.6] Hartshorne proved that a vector bundle E of rank two over X is ample if and only if the following two conditions hold:

(1) $\text{degree}(E) > 0$, and
(2) $\text{degree}(Q) > 0$ for every quotient line bundle Q of E.

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If the field k is allowed to be of positive characteristic, then the above criterion fails. Indeed, in case of positive characteristics there are examples of rank two vector bundle E of positive degree such that all the quotient line bundles of E are of positive degree while E is not ample ([4, p. 84, §3], [2]). The right generalization of [3, Corollary 7.6] in case of positive characteristics was obtained by Lange in [5] which we will now describe.

Now let k be an algebraically closed field of characteristic p, with $p > 0$. Let X be a connected smooth projective curve defined over k. Let $F_X : X \to X$ be the Frobenius morphism of X. For any rank two vector bundle V over X, following [5, p. 197] define $s(V) := \text{degree}(V) - 2\,\text{degree}(L)$, where L is a line subbundle of V of maximal degree.

Take any vector bundle E over X of rank two. In [5, p. 201, Proposition 4.4] the following criterion for E to be ample was proved:

(1) if E is not stable or $(F_X^n)^*E$ is stable for all $n \geqslant 1$, where F_X^n is the n-fold iteration of the Frobenius morphism F_X, then E is ample if and only if $\text{degree}(E) > 0$ and $\text{degree}(Q) > 0$ for every quotient line bundle Q of V;
(2) if there is an integer n such that $(F_X^{n-1})^*E$ is stable but $(F_X^n)^*E$ is not stable, then E is ample if and only if $\text{degree}(E) > -s((F_X^n)^*E)/p^n$.

Take any vector bundle E over X of arbitrary rank. For any integer $n \geqslant 1$, let E_n denote the final quotient in the Harder–Narasimhan filtration of $(F_X^n)^*E$. So E_n is a semistable quotient bundle of $(F_X^n)^*E$, and $E_n = (F_X^n)^*E$ if $(F_X^n)^*E$ is semistable. Consider the sequence of rational numbers

$$\mu_{\min}(E,n) := \frac{\text{degree}((F_X^n)^*E)}{\text{rank}((F_X^n)^*E)}.$$

There is an integer $n(E)$ such that

$$\frac{\mu_{\min}(E,n(E))}{p^{n(E)}} = \frac{\mu_{\min}(E,n(E)+j)}{p^j}$$

for all $j \geqslant 1$, where p is the characteristic of the field k. The above equality is deduced using [6, p. 259, Theorem 2.7]; see Section 2 for the details. This number $\mu_{\min}(E,n(E))/p^{n(E)}$ at which the sequence of numbers $\{\mu_{\min}(E,n)/p^n\}_{n=1}^{\infty}$ stabilizes will be denoted by $\overline{\mu}_{\min}(E)$.

Our aim here is to prove the following criterion for a vector bundle over X to be ample.

Theorem 1.1. *A vector bundle E over X is ample if and only if $\overline{\mu}_{\min}(E) > 0$.*

When $\text{rank}(E) = 2$, it is easy to see that Theorem 1.1 is equivalent to the earlier mentioned criterion in [5]. Indeed, if $(F_X^{n-1})^*E$ is stable but $(F_X^n)^*E$ is not stable, then $\text{degree}(E) + s((F_X^n)^*E)/p^n = 2\overline{\mu}_{\min}(E)$.

In [1] it was shown that an ample vector bundle E over X has the following property: the vector bundle $(F_X^n)^*E$ is globally generated provided the integer n is sufficiently large.

2. Criterion for amplitude

Let k be an algebraically closed field of characteristic p, with $p > 0$. Let X be an irreducible smooth projective curve defined over k. Let

$$F_X : X \to X \tag{1}$$

be the Frobenius morphism of X. For any $n \geqslant 1$, let $F_X^n : X \to X$ be the n-fold iteration of the self-morphism F_X. For notational convenience F_X^0 will denote the identity morphism of X.

A vector bundle V over X is called *semistable* if

$$\mu(W) := \frac{\text{degree}(W)}{\text{rank}(W)} \leqslant \frac{\text{degree}(V)}{\text{rank}(V)} =: \mu(V)$$

for every subbundle $W \subset V$. Furthermore, V is called *strongly semistable* if the pullback $(F_X^n)^* V$ is semistable for all $n \in \mathbb{N}$, where F_X is the Frobenius morphism in (1).

Lemma 2.1. *Let E be a strongly semistable vector bundle over X with* $\text{degree}(E) > 0$. *Then the vector bundle E is ample.*

Proof. Fix an integer n_0 such that

$$\mu(E) := \frac{\text{degree}(E)}{\text{rank}(E)} > \frac{2g_X}{n_0 p}, \tag{2}$$

where g_X is the genus of X and p is the characteristic of the base field k. Fix a closed point $x_0 \in X$. Set

$$E' := \mathcal{O}_X(-x_0) \otimes_{\mathcal{O}_X} (F_X^{n_0})^* E. \tag{3}$$

We will show that this vector bundle E' is generated by its global sections.

Take any closed point $x \in X$, and consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} E' \to E' \to E'_x \to 0$$

over X. This gives a long exact sequence of cohomologies

$$\cdots \to H^0(X, E') \to E'_x \to H^1(X, \mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} E') \to \cdots. \tag{4}$$

Since E is strongly semistable, the vector bundle E' is semistable. Hence the vector bundle

$$E'' := \big(\mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} E'\big)^* \otimes K_X \tag{5}$$

is semistable, where K_X is the canonical line bundle of X. On the other hand, using (2) we have

$$\mu(E'') = 2g_X - 1 - \mu(E') = 2g_X - 1 - \mu(E) n_0 p + 1 = n_0 p \left(\frac{2g_X}{n_0 p} - \mu(E) \right) < 0,$$

where E' and E'' are defined in (3) and (5) respectively. Since the vector bundle E'' is semistable of negative degree, we have $H^0(X, E'') = 0$ which, using Serre duality, implies that $H^1(X, \mathcal{O}_X(-x) \otimes_{\mathcal{O}_X} E') = 0$. Now from (4) it follows that the vector bundle E' is globally generated.

Since $(F_X^{n_0})^*E = E' \otimes_{\mathcal{O}_X} \mathcal{O}_X(x_0)$ (recall (3)) with E' globally generated and $\mathcal{O}_X(x_0)$ ample, it follows that the vector bundle $(F_X^{n_0})^*E$ is ample [3, p. 67, Corollary 2.3]. The morphism F_x being surjective, this implies that the vector bundle E is ample [3, p. 73, Proposition 4.3]. This completes the proof of the lemma. $\square$

Any vector bundle V over X admits a unique filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$$

of subbundles (ℓ depends on V), known as the *Harder–Narasimhan* filtration, where each successive quotient V_i/V_{i-1}, $i \in [1, \ell]$, is semistable and $\mu(V_j/V_{j-1}) > \mu(V_{j+1}/V_j)$ for all $j \in [1, \ell - 1]$. For any integer $n \geqslant 0$, let

$$0 = V_0^n \subset V_1^n \subset \cdots \subset V_{\ell_n}^n = (F_X^n)^*V \tag{6}$$

be the Harder–Narasimhan filtration of the vector bundle $(F_X^n)^*V$.

There is an integer m (it depends on V and it is not unique) such that all the successive quotient bundles V_i^m/V_{i-1}^m, $i \in [1, \ell_m]$, in (6) are strongly semistable [6, p. 259, Theorem 2.7]; see [6, p. 258, §2.6] for the definition of fdHN bundle mentioned in Theorem 2.7 of [6]. This immediately implies that $\ell_{m+j} = \ell_m$ for all $j \geqslant 1$, where ℓ_{m+j} are as in (6), and furthermore, $V_i^{m+j} = (F_X^j)^*V_i^m$ for all $j \in [1, \ell_m]$.

Set

$$\mu_{\min}(V, n) := \mu\big(\big((F_X^n)^*V\big)/V_{\ell_n-1}^n\big),$$

where $V_{\ell_n-1}^n$ is as in (6). We noted above that $V_{\ell_n-1}^{m+j} = (F_X^j)^*V_{\ell_n-1}^m$ for all $j \geqslant 1$. Therefore, we have

$$\mu_{\min}(V, m + j) = p^j \mu_{\min}(V, m), \tag{7}$$

where p is the characteristic of k (this follows from the fact that the degree of F_X is p).

Set

$$\overline{\mu}_{\min}(V) := \frac{\mu_{\min}(V, m)}{p^m}, \tag{8}$$

where m is as above. From (7) it follows immediately that $\overline{\mu}_{\min}(V)$ does not depend on the choice of m.

Theorem 2.2. *A vector bundle E over X is ample if and only if $\overline{\mu}_{\min}(E) > 0$, where $\overline{\mu}_{\min}(E)$ is defined in* (8).

Proof. Assume that E is ample. The morphism F_X being finite, this implies that the vector bundle $(F_X^n)^*E$ is ample for all $n \geqslant 1$ [3, p. 73, Proposition 4.3]. Therefore, the quotient bundle $(F_X^n)^*E/E_{\ell_n-1}^n$ is ample [3, p. 66, Proposition 2.2], where

$$0 = E_0^n \subset E_1^n \subset \cdots \subset E_{\ell_n}^n = (F_X^n)^*E \tag{9}$$

is the Harder–Narasimhan filtration of $(F_X^n)^*E$. Since the degree of any ample vector bundle over X is positive [3, p. 68, Corollary 2.6], we conclude that $\overline{\mu}_{\min}(E) > 0$.

To prove the converse assume that $\overline{\mu}_{\min}(E) > 0$. Take a sufficiently large integer n. Consider the Harder–Narasimhan filtration (9) of $(F_X^n)^*E$. We noted from [6] that for each $i \in [1, \ell_n]$ the quotient E_i^n/E_{i-1}^n in (9) is a strongly semistable vector bundle. Also,

$$\mu(E_i^n/E_{i-1}^n) \geqslant \mu(E_{\ell_n}^n/E_{\ell_n-1}^n) = \overline{\mu}_{\min}(E) > 0$$

for all $i \in [1, \ell_n]$. Hence using Lemma 2.1 we conclude that the vector bundle E_i^n/E_{i-1}^n is ample for each $i \in [1, \ell_n]$. Since the filtration (9) of $(F_X^n)^*E$ has the property that each successive quotient is an ample vector bundle, it follows that the vector bundle $(F_X^n)^*E$ is ample [3, p. 71, Corollary 3.4]. As the morphism F_X is surjective, this implies that the vector bundle E is ample [3, p. 73, Proposition 4.3]. This completes the proof of the theorem.

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